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TWO-LOOP QUARK SELF-ENERGY IN A NEW FORMALISM (I) OVERLAPPING DIVERGENCES

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ABSTRACT

A new integration technique for multi-loop Feynman integrals, called the *matrix method*, is developed and then applied to the divergent part of the overlapping two-loop quark self-energy function $i\Sigma$ in the light-cone gauge $n{\cdot}A^a(x) = 0$, $n^2 = 0$. It is shown that the coefficient of the double-pole term is strictly local, even off mass-shell, while the coefficient of the single-pole term contains local as well as nonlocal parts. On mass-shell, the single-pole part is local, of course. It is worth noting that the original overlapping self-energy integral reduces eventually to 10 covariant and 38 noncovariant-gauge integrals. We were able to verify explicitly that the *divergent parts* of the 10 double covariant-gauge integrals agreed precisely with those currently used to calculate radiative corrections in the Standard Model.

Our new technique is amazingly powerful, being applicable to massive and massless integrals alike, and capable of handling both covariant-gauge integrals and the more difficult noncovariant-gauge integrals. Perhaps the most important feature of the matrix method is the ability to execute the 4ω-dimensional momentum integrations in a single operation, exactly and in analytic form. The method works equally well for other axial-type gauges, notably the temporal gauge ($n^2 > 0$) and the pure axial gauge ($n^2 < 0$).

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1. Introduction

Traditional perturbation theory with its emphasis on Feynman diagrams continues to play a central role in quantum field theory. The success of the perturbative approach hinges decisively on the accurate computation of multi-loop Feynman integrals. Of course, there exists a vast variety of such integrals, but here we shall only distinguish between covariant-gauge Feynman integrals and noncovariant-gauge integrals. As the name implies, covariant-gauge Feynman integrals occur in theories quantized in a covariant gauge, such as the Fermi gauge or the unitary gauge. Noncovariant-gauge integrals, by contrast, arise whenever a noncovariant gauge is implemented, such as the powerful light-cone gauge [1] or the infamous Coulomb gauge [2]. While both types of integrals are known to require patience and a healthy respect for detail, it is also true that noncovariant integrals remain as popular as skunks at a garden party, as seen from the small fraction of higher-loop calculations in QCD and Yang-Mills theory [3-8].

According to the literature, multi-loop integrals, such as $\int d^{2\omega}q \int d^{2\omega}k\, f(q,k)$, have been evaluated almost exclusively by the *nested method* [9-14], in which the four-momentum integrations are carried out sequentially (initial exponential parametrization of the propagators is assumed). For instance, for the double integral mentioned above, one first integrates over $d^{2\omega}k$, then over $d^{2\omega}q$ (or conversely). Dimensional regularization will be used throughout this paper, with 2ω denoting the dimensionality of complex space-time. We have no intention of reviewing here the dominant characteristics and various idiosyncrasies of the nested approach, except to say that it has been used and abused with varying degrees of success.

Instead, we should like to propose an alternative procedure to the nested method, called the *matrix integration technique*, in which the two momentum

integrals in $\int d^{2\omega}q \int d^{2\omega}k\, f(q,k)$ are integrated over 4ω-dimensional space in a *single operation.* Perhaps the most appealing feature of this method is the ability to perform the momentum integration exactly and in closed form, thereby guaranteeing from the outset a certain amount of what might be called "calculational streamlining". In the course of our investigation we shall encounter several examples of this "streamlining". It turns out that the matrix technique works for covariant and noncovariant gauges alike, and regardless whether the integrals are massive or massless.

The purpose of our integration program may now be stated as follows:

1. To develop for two-loop integrals an alternative approach to the nested method, called the *matrix integration technique.*
2. To apply this technique to the complete two-loop quark self-energy function in the light-cone gauge $n\cdot A^a(x) = 0$, $n^2 = 0$.
3. To derive, if possible, the necessary counterterms and use these to renormalize the theory to two-loop order.
4. To extend the matrix technique to other axial gauges, notably to the temporal gauge defined by $n\cdot A^a(x) = 0$, $n^2 > 0$, and to the pure axial gauge $n\cdot A^a(x) = 0$, $n^2 < 0$.

As indicated above, the testing ground for our matrix technique will be the quark self-energy function to two loops, depicted graphically in Figures 1 and 2. For pedagogical reasons, we have decided to report our results in two separate papers. In paper I, we shall discuss various mathematical tools and then apply these to compute the divergent portion of the *overlapping* quark self-energy (Fig. 1). The remaining two-loop graphs, including the non-trivial rainbow diagram (Fig. 2(a)), will be treated in paper II, where we shall also study the counterterms required for renormalization.

The plan of paper I is as follows. In Section 2 we define the structure of the overlapping quark self-energy function and review some standard formulas needed for one-loop calculations. In Section 3, we summarize the main features of the matrix method, including reduction of the integrand to Gaussian form and the evaluation of 4ω-dimensional Gaussian integrals. The parameter integrations are carried out in Sections 4 and 5. But in order to prepare the reader for certain technical subtleties, we shall first make a short detour to introduce some terminology.

For each factor in the denominator of the integrand of $\int d^{2\omega}q \int d^{2\omega}k\, f(q,k)$, there will be a *Schwinger parameter*, α_j say $(j = 1, 2, \ldots)$, with an infinite domain of integration; i.e., $\alpha_j \,\epsilon\, [0, \infty]$. For the overlapping diagram discussed in this article, we transform the set of parameters $\{\alpha_j\}$ to a more "user-friendly" set S, containing two types of parameters: one *type I* parameter (A), with an infinite domain, and up to six *type II* parameters $(\lambda, \beta, G, b, h, a)$, with finite domains. We shall demonstrate in Section 4 that integration over the lone type I parameter leads to a simple pole, and in Section 5 that one more simple pole (and hence a double pole overall) emerges from integration over one of the finite type II parameters. The paper concludes in Section 6 with a short discussion.

2. Basic Tools

(a) Notation and review of the light-cone prescription

In Yang-Mills theory, the light-cone gauge is characterized by

$$n^\mu A^a_\mu(x) = 0, \qquad n^2 = 0, \qquad \mu = 0, 1, 2, 3, \tag{2.1}$$

where $n_\mu = (n_0, \vec{n})$ defines a fixed axis in four-space [1]. We use a $(+, -, -, -)$ metric and employ dimensional regularization in a space-time of 2ω dimensions.

The relevant SU$(3)_c$ color Lagrangian density reads [15]

$$\mathcal{L} \;=\; \mathcal{L}_{int} - \lim_{\lambda\to 0} \tfrac{1}{2\lambda}(n\cdot A^a)^2, \qquad n^2 = 0, \tag{2.2}$$

$$\mathcal{L}_{int} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_k i\overline{\psi}^k_\alpha(\gamma_\mu D^\mu_{\alpha\beta} + m_k\delta_{\alpha\beta})\psi^k_\beta\,, \tag{2.3}$$

where ψ_α and A^a_μ represent fermion and gluon fields, respectively, γ_μ are 4×4 Dirac gamma-matrices, $a = 1, 2, \ldots, 8$ is the group index, α, β are color indices, λ is the gauge parameter, $k = u, d, \ldots$ is the quark flavor index, and m_k are quark rest masses. Moreover,

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2.4a}$$

is the field strength (g is the QCD coupling constant), and

$$D_{\alpha\beta,\mu} \;=\; \delta_{\alpha\beta}\partial_\mu - igT^a_{\alpha\beta}A^a_\mu \tag{2.4b}$$

denotes the covariant derivative ($\partial_\mu \equiv \partial/\partial x^\mu$). $T^a_{\alpha\beta}$ are the generators of the gauge group SU(3) which obey the commutation relations

$$[T^a, T^b] \;=\; if^{abc}T^c, \tag{2.4c}$$

f^{abc} being antisymmetric structure constants. In the light-cone gauge, the Lagrangian density (2.2) leads to the gluon propagator

$$G^{ab}_{\mu\nu}(q) \;=\; \frac{-i\,\delta^{ab}}{q^2 + i\epsilon}\left(g_{\mu\nu} - \frac{q_\mu n_\nu + q_\nu n_\mu}{q\cdot n}\right), \qquad \epsilon > 0. \tag{2.5}$$

The spurious poles of the factor $(q\cdot n)^{-1}$ will be handled by the n^*_μ-prescription [16-17]:

$$\frac{1}{q\cdot n} \;=\; \lim_{\epsilon\to 0} \frac{q\cdot n^*}{q\cdot n\; q\cdot n^* + i\epsilon}\,, \qquad \epsilon > 0, \tag{2.6}$$

with $n_\mu = (n_0, \vec{n})$ and $n^*_\mu = (n_0, -\vec{n})$.

(b) The overlapping quark self-energy

Application of the Feynman rules for QCD, with the quark-quark-gluon vertex factor $igT^a_{\alpha\beta}\gamma^\mu$, the quark propagator $i/(\not{q}-m)$, and the gluon propagator (2.5), yields the following expression for the two-loop overlapping quark self-energy function in the light-cone gauge (Fig. 1):

$$I_0 = \; ig^4 T^b_{DA} T^a_{CD} T^b_{BC} T^a_{AB} I, \tag{2.7}$$

$$I = \int_M \int_M \frac{d^4q d^4k}{(2\pi)^8} \frac{\gamma^\mu\ (\not{p}-\not{q}+m)\ \gamma^\tau\ (\not{p}-\not{k}-\not{q}+m)\ \gamma^\nu\ (\not{p}-\not{k}+m)\ \gamma^\sigma}{q^2\,[(p-q)^2-m^2]\ [(p-k-q)^2-m^2]\ [(p-k)^2-m^2]\ k^2} \times \left(g_{\mu\nu} - \frac{n_\mu q_\nu + n_\nu q_\mu}{n\cdot q}\right)\left(g_{\sigma\tau} - \frac{n_\sigma k_\tau + n_\tau k_\sigma}{n\cdot k}\right), \tag{2.8}$$

where m is the quark mass, p the quark 4-momentum, $\int_M$ denotes integration over Minkowski space, and where the five $i\epsilon$ terms have been omitted. Evaluation of the integral I by means of the matrix method is the main objective of this article.

To begin with, we observe that I in Eq. (2.8) diverges at the following three types of limits:

(i) $|q|$ and/or $|k|$ $\to \infty$;

(ii) q and/or k $\to$ zeros of the quadratic factors in the denominator of Eq. (2.8);

(iii) $q\cdot n \to 0$ and/or $k\cdot n \to 0$.

The first two types of divergence are handled by dimensional regularization and Wick rotation. The third type of divergence (iii) is symptomatic of axial-type gauges, and requires application of the n^*_μ-prescription (2.6) to the spurious poles of $(q\cdot n)^{-1}$ and $(k\cdot n)^{-1}$. Since the n^*_μ-prescription permits a Wick rotation ($iq_4 = q_0$, $ik_4 = k_0$, $ip_4 = p_0$), we may Wick-rotate the entire integrand from

Minkowski space to Euclidean space.

Next we apply exponential parametrization to every quadratic factor F_j $(j = 1, 2, \ldots)$ in the denominator of the rotated integrand:

$$\frac{1}{F_j} \;=\; \int_0^\infty d\alpha_j \;\exp\left(-\alpha_j F_j\right), \qquad F_j > 0\,, \tag{2.9}$$

thereby replacing the product of quadratic factors in the denominator by a *sum* of quadratic terms in the exponent. Completing the square in this exponent, we can then integrate over the momentum variables by means of the generalized Gaussian formula:

$$\int d^{2\omega}q \;\exp\left(-Aq^2\right) \;=\; \left(\frac{\pi}{A}\right)^{\omega}, \qquad A \,>\, 0. \tag{2.10}$$

(c) Two-loop integrations: the traditional approach

Although the methods described in Section 2(b) are applicable to any multi-loop integral, the ever-increasing technical difficulties have drastically restricted the number of explicit computations. Here is a brief look at some typical problems, encountered already in double integrals such as $\int d^{2\omega}q \int d^{2\omega}k\, f(q,k)$.

Two-loop integrals have traditionally been computed by the so-called *nested method.* As alluded to in the Introduction, the basic idea behind the nested method is to integrate over the loop momenta one loop at a time, i.e., to integrate over k as completely as possible, before attempting integration over q. In many two-loop cases the k-integral is so complicated, however, that we cannot fully compute it, unless we express the result in the form of a series, such as the Laurent series $\sum_j (\omega-2)^j\, T_j(n,p,q;m)$. Each of the functions T_j must then be multiplied by the remaining q-dependent factors in the original integrand, and the resulting

expressions integrated over 2ω-dimensional q-space. One hopes, of course, that only the first few (i.e., small j) terms of the Laurent series will contribute to the pole parts of the final result. Unfortunately, no such luck prevails in general, as may be seen from the following typical, albeit simplified, example:

$$\int_1^\infty q^{1-\omega}\, q^{2\omega-4}\, dq \;\rightarrow\; \sum_{j=0}^{\infty} \left(\frac{(2\omega-4)^j}{j!} \int_1^\infty q^{1-\omega}\, (\ln q)^j\, dq \right). \tag{2.11}$$

Integration of the j-th term on the right-hand side by parts j times yields $\sum_j 2^j$, whereas direct evaluation of the left-hand side yields $(2-\omega)^{-1}$. The inconsistency stems from the fact that the exponential series for $q^{2\omega-4}$ does *not converge uniformly* as $|q| \rightarrow \infty$.

In order to improve the behaviour for large values of $|q|$, we could try to express the result of the k-integration as a series in powers of $1/q^2$. One may always obtain such a series by applying the binomial formula to the integrand *before* integrating over the last one or two Schwinger parameters associated with the k-integration. On the surface, this strategy looks promising, but on closer inspection we sometimes find that the resulting series *fails to converge* for certain combinations of values of q and the unintegrated Schwinger parameters. Some of the UV divergences of the integral I, for instance, occur precisely in regions where the binomial series diverges.

These simple examples serve as a potent reminder that (1) the divergent terms for overlapping loops do not arise merely from the divergent and finite terms ($j \leq 0$) of the individual loops, and that (2) care must be exercised in choosing series which converge uniformly over the entire region of integration. We shall see in Sections 4 and 5 that point (2) is also crucial in the context of the matrix method, where similar convergence issues arise, albeit in vastly simplified form.

3. The Matrix Method

(a) Basics

In this section we propose an alternative procedure to the nested method, called the matrix integration technique, or *matrix method* for short. In the matrix method we regard q-space and k-space as subspaces of a single 4ω-dimensional momentum space and proceed as in the case of one-loop integrals. In other words, we Wick-rotate q_0 and k_0, use the n^*_μ-prescription (2.6) for $(q{\cdot}n)^{-1}$ and $(k{\cdot}n)^{-1}$, and apply the exponential parametrization (2.9) to all denominator factors. The two-loop integral I in Eq. (2.8) for the overlapping diagram in Figure 1 then assumes the form:

$$\int_E d^{2\omega}q \int_E d^{2\omega}k \ f(q,k) \ = \ \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \ \ldots \ J[P(q,k)] \qquad \equiv \ I_E[f]\,, \tag{3.1}$$

where:

$$f(q,k) \ = \ \frac{i^2(-1)^4 \ \gamma^\mu \ (\not p - \not q - m) \ \gamma^\tau \ (\not p - \not k - \not q - m) \ \gamma^\nu \ (\not p - \not k - m) \ \gamma^\sigma}{(2\pi)^{4\omega} \ q^2 \, [(p-q)^2 + m^2] \ [(p-k-q)^2 + m^2] \ [(p-k)^2 + m^2] \ k^2}$$

$$\times \ \frac{(n{\cdot}q\delta_{\mu\nu} - n_\mu q_\nu - n_\nu q_\mu)}{n{\cdot}q} \, \frac{(n{\cdot}k\delta_{\sigma\tau} - n_\sigma k_\tau - n_\tau k_\sigma)}{n{\cdot}k} \qquad \text{(in this case)}; \tag{3.2}$$

$$J[P(q,k)] \ \equiv \int_E d^{2\omega}q \int_E d^{2\omega}k \ P(q,k) \ \exp(-\,\mathbf{z}^\top \mathbf{M}\mathbf{z} \ + \ 2\mathbf{B}^\top \mathbf{z} \ - \ C)\,; \tag{3.3}$$

$P(q,k) \ \equiv$ the numerator of f, multiplied by $-q{\cdot}n^*/n_0^2$ and/or $-\,k{\cdot}n^*/n_0^2$ from the n^*_μ-prescription (as applicable);

$\mathbf{z} \ \equiv \ (k_4,\, q_4,\, k_3,\, q_3,\, \ldots)^\top \ = $ a 4ω-component column-vector ($^\top \equiv$ transpose); $\mathbf{M}$ is a $4\omega \times 4\omega$ matrix; $\mathbf{B}$ is a 4ω-component column-vector; C is a scalar; and $\int_E$ denotes integration over Euclidean space. Since $\mathbf{M}_{ij}$ is the coefficient of $z_i z_j$ in the exponent of the integrand in Eq. (3.3), we may always symmetrize $\mathbf{M}$

("z_i" refers to the i^{th} component of $\mathbf{z}$).

In the case of the integral I, with f given by Eq. (3.2), let us take $\alpha_1, \ldots, \alpha_7$ to be the parameters corresponding to the factors $n{\cdot}q$, q^2, $[(p-q)^2+m^2]$, $[(p-k-q)^2+m^2]$, $[(p-k)^2+m^2]$, k^2, and $n{\cdot}k$, respectively. We then find that $\mathbf{M}$, $\mathbf{B}$, and C take particularly simple forms when we change the variables of integration from $\{\alpha_1, \ldots, \alpha_7\}$ to the more "user-friendly" set $S = \{A;\, \lambda, \beta, G, b, h, a\}$, where

$$\left.\begin{array}{rlrl} & A \equiv\ \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7\,, & & \\ G \equiv & \alpha_4/A\,, & b \equiv & (\alpha_4+\alpha_5)/A\,, \\ \beta \equiv & (\alpha_3+\alpha_4)/A\,, & h \equiv & (\alpha_4+\alpha_5+\alpha_6)/A\,, \\ \lambda \equiv & (\alpha_2+\alpha_3+\alpha_4)/A\,, & a \equiv & (\alpha_4+\alpha_5+\alpha_6+\alpha_7)/A\,. \end{array}\right\} \tag{3.4}$$

As explained in the Introduction, A is a type I parameter with an infinite domain, while $\lambda, \beta, G, b, h, a$ are type II parameters with finite domains. In terms of these parameters,

$$\left.\begin{array}{l} \mathbf{M} = A\begin{bmatrix} a & G & & & & & & \\ G & 1-a & & & & & & \\ & & a & G & & & & \\ & & G & 1-a & & & & \\ & & & & h & G & & \\ & & & & G & \lambda & & \\ & & & & & & h & G \\ & & & & & & G & \lambda \end{bmatrix}, \quad \mathbf{B} = A\begin{bmatrix} bp_4 \\ \beta p_4 \\ bp_3 \\ \beta p_3 \\ bp_2 \\ \beta p_2 \\ bp_1 \\ \beta p_1 \end{bmatrix}, \\ C = A\,(b+\beta-G)\,(p^2+m^2)\,, \end{array}\right\} \tag{3.5}$$

where $n_1 = n_2 = 0$ for simplicity. Note that the lower half of $\mathbf{M}$ has $2\omega-2$ pairs of rows, and the right-hand half $2\omega-2$ pairs of columns, so that

$$\sqrt{\det(\mathbf{M})} \;=\; A^{2\omega} D_\parallel\, D_\perp^{\omega-1}\,, \tag{3.6}$$

$$\text{where} \qquad D_\parallel \equiv a(1-a)-G^2\,, \qquad D_\perp \equiv \lambda h - G^2\,. \tag{3.7}$$

(b) Momentum Integration

Our first major task is to evaluate the momentum-space integral $J[P]$ in Eq. (3.3), P being a polynomial in the components of q and k. Differentiating Eq. (3.3) partially with respect to B_i, we obtain

$$\frac{\partial J[P]}{\partial B_i} \;=\; 2\, J[z_i P]. \tag{3.8}$$

Once $J[1]$ is known, we can derive $J[P]$ for any polynomial P by repeated application of formula (3.8).

To evaluate $J[1]$, we first diagonalize the quadratic form in the exponent of the integrand in Eq. (3.3). Since $\mathbf{M}$ is symmetric, there exists a matrix $\mathbf{R}$ such that $\mathbf{R}^\top\mathbf{R} = 1$, and $\mathbf{R}\mathbf{M}\mathbf{R}^\top$ is diagonal. Letting $\mathbf{z} = \mathbf{R}^\top\mathbf{y} + \mathbf{M}^{-1}\mathbf{B}$ in Eq. (3.3), and choosing $P = 1$, we obtain

$$J[1] \;=\; \exp\left(\mathbf{B}^\top\mathbf{M}^{-1}\mathbf{B} \;-\; C\right) \int_E \exp\left(-\mathbf{y}^\top\mathbf{R}\mathbf{M}\mathbf{R}^\top\mathbf{y}\right) d^{4\omega}\mathbf{y}\,. \tag{3.9}$$

With $\mathbf{R}\mathbf{M}\mathbf{R}^\top$ diagonal, the above integral is just a product of one-dimensional Gaussian integrals, so that

$$J[1] \;=\; \frac{\pi^{2\omega}\;\exp\left(\mathbf{B}^\top\mathbf{M}^{-1}\mathbf{B} - C\right)}{\sqrt{\det(\mathbf{M})}}\;, \tag{3.10}$$

$\det(\mathbf{R})$ and $\det(\mathbf{R}^\top)$ having cancelled each other. Applying formula (3.8) repeatedly to Eq. (3.10), we get

$$J[z_i] \;=\; m_i\, J[1]\,, \tag{3.11a}$$

$$J[z_i z_j] \;=\; [m_i m_j + (\tfrac{\mathbf{M}^{-1}}{2})_{ij}]\, J[1]\,, \tag{3.11b}$$

$$J[z_i z_j z_k] \;=\; [m_i m_j m_k + m_i(\tfrac{\mathbf{M}^{-1}}{2})_{jk} + m_j(\tfrac{\mathbf{M}^{-1}}{2})_{ik} + m_k(\tfrac{\mathbf{M}^{-1}}{2})_{ij}]\, J[1]\,, \tag{3.11c}$$

$$\vdots$$

where $\mathbf{m} \equiv \mathbf{M}^{-1}\mathbf{B}$, and $\partial m_i/\partial B_j = (\mathbf{M}^{-1})_{ij}$.

When $\mathbf{M}$ and $\mathbf{B}$ are given by Eqs. (3.5), $D_{\|}$ and $D_{\perp}$ by Eqs. (3.7), and $p_{\|}$ and $p_{\perp}$ by

$$p_{\|} \equiv (0,\ 0,\ p_3,\, p_4) \ = \ \frac{n^{*}\!\cdot p\ n + n\!\cdot\! p\ n^{*}}{n^{*}\!\cdot n}\,, \tag{3.12a}$$

$$p_{\perp} \equiv (p_1,\, p_2,\ 0,\ 0) \ = p - p_{\|}\,, \tag{3.12b}$$

we find

$$\mathbf{m} = (r_4,\, s_4,\, r_3,\, s_3,\, \ldots)^{\top}, \tag{3.13}$$

with

$$r = \ r_{\|} + r_{\perp} \ = \ \left(\frac{(1-a)b - G\beta}{D_{\|}}\right) p_{\|} + \left(\frac{\lambda b - G\beta}{D_{\perp}}\right) p_{\perp}\,, \tag{3.14a}$$

$$s = \ s_{\|} + s_{\perp} \ = \ \left(\frac{a\beta - Gb}{D_{\|}}\right) p_{\|} + \left(\frac{h\beta - Gb}{D_{\perp}}\right) p_{\perp}\,. \tag{3.14b}$$

(For the sake of clarity, we have dropped the Lorentz indices on $p_{\|}$, $p_{\perp}$, n, n^{*}, r, $r_{\|}$, $r_{\perp}$, s, $s_{\|}$, and $s_{\perp}$.) Substituting from Eqs. (3.5) and (3.13) into Eqs. (3.10) and (3.11), and exploiting the linearity of $J[P]$, we obtain the useful momentum-space integrals given in Appendix A.

The off-diagonal entries in $\mathbf{M}$ originate from the $q\cdot k$ term in the exponent of the integrand in Eq. (3.3). In the integral I, this term comes, of course, from the parametrization of the denominator factor $[(p-k-q)^2+m^2]$, which in turn comes from the internal line shared by the two loops in Figure 1. Note that in the *absence* of such a shared line, the integral I would factor into two separate one-loop integrals. By diagonalizing $\mathbf{M}$, it might appear that we have effectively disentangled the overlapping loops, but that is not the case. All we have done is shift the problem to the parameter integrations. To see this, we observe that the integrands of these integrations still contain the off-diagonal elements of $\mathbf{M}$ by virtue of the factors $\mathbf{M}^{-1}$ and $\sqrt{\det(\mathbf{M})}$ in Eqs. (3.10) and (3.11). Parameter integrations will be discussed in Sections 4 and 5.

(c) Reduction of the integrand

We observe in Eqs. (3.11) that the complexity of J increases dramatically with the degree of its argument P, P being the numerator of the integrand f in Eq. (3.1) multiplied by $-q\cdot n^*/n_0^2$ and/or $-k\cdot n^*/n_0^2$ from the n_μ^*-prescription if applicable. It is desirable, therefore, to reduce the degree of P as much as possible before integration. For I, the initial degree of P is 7, but we can reduce this number to 3 by executing the following trivial operations before Wick-rotation:

1. We completely expand the numerator of the integrand in Eq. (2.8) into a sum of products. We then drop all terms which are shown by power counting to be UV-convergent, since we are only interested in the divergent terms. We recall that the n_μ^*-prescription is consistent with power counting [1], and that Weinberg's Theorem asserts that a two-loop integral is UV-convergent if the net power of the integrand in each of q, k, and both q and k together, is less than zero [18]. For I, the result is that terms containing more than one factor of m and/or $\not{p}$ may be dropped.

2. In the remaining terms, we rearrange the non-commuting Dirac matrix factors with the help of the relation $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$, so that most terms with more than one matrix factor cancel with one another. Some terms involving $\not{q}\not{k}\not{n}$ remain, but in these terms we may replace $\not{q}\not{k}$ by $\frac{1}{2}(\not{q}\not{k}+\not{k}\not{q}) = q\cdot k$ because of the symmetry between q and k in Eq. (2.8). Also note that $\gamma^\mu\gamma_\mu = 2\omega$, and that $n^2 = 0$ in the light-cone gauge.

3. In each of the remaining terms, we cancel as many factors as possible with factors in the denominator of the integrand. (In some cases, identities such as $2q\cdot k \;=\; p^2 - m^2 - [(p-q)^2 - m^2] + [(p-k-q)^2 - m^2] - [(p-k)^2 - m^2]\,,$ $2p\cdot q \;=\; p^2 - m^2 - [(p-q)^2 - m^2] + q^2\,,$ etc. are helpful.) The resulting

integrands will have fewer denominator factors than were present originally in Eq. (2.8), but Eqs. (3.5) to (3.14) are still valid, provided we set the α parameters which correspond to absent factors to zero in Eqs. (3.4), and omit the integrations over these parameters from Eq. (3.1). (We see from Eq. (2.9) that this procedure is equivalent to inserting factors of exp(0) in the integrand in Eq. (3.3). The advantage of this strategy is obvious: it allows us to use one common expression for J for all integrals with the same P, even though the integrands of all these integrals will have different denominators.)

4. Finally, we reduce the number of independent terms by interchanging q and k in selected terms.

The above four steps were carried out by means of a computer program designed by one of the authors (J.W.). Application of these steps, followed by Wick-rotation, transforms the integral I into the sum of 53 integrals, represented by the 53 non-blank entries in Tables 1 to 6. Each of the 53 integrals is equal to an entry in one of the tables times the Euclidean-vector expression at the left-hand side of its row of the table, divided by $(2\pi)^{4\omega}$ times the denominator at the bottom of the table, and integrated over 4ω-dimensional qk-space.

The symbols n, q, Q, $\wedge$, K, and k at the bottoms of the tables represent denominator factors in accordance with the following scheme:

$$\begin{array}{ccccccc} \mathrm{n} & \mathrm{q} & \mathrm{Q} & \wedge & \mathrm{K} & \mathrm{k} & \mathrm{n} \\ n\cdot q & q^2 & [(p-q)^2+m^2] & [(p-k-q)^2+m^2] & [(p-k)^2+m^2] & k^2 & n\cdot k \end{array} \tag{3.15}$$

Thus, an "n" on the left side of the denominator represents $n\cdot q$, while an "n" on the right represents $n\cdot k$. For example, the entry $4-4\epsilon$ in the fifth row and second-last column of Table 1 corresponds to the Euclidean-space integral

$$\frac{(4-4\epsilon)\; n\cdot p}{(2\pi)^{4\omega}} \int_E \int_E \frac{\not{q}\;\; d^{2\omega}q\; d^{2\omega}k}{n\cdot q\; q^2\,[(p-q)^2+m^2]\;[(p-k-q)^2+m^2]\;k^2}\,, \tag{3.16}$$

where $\epsilon \equiv 2-\omega\,;$ ϵ occurs in the integrand only because $\gamma^\mu\gamma_\mu = 2\omega$ (see step 2 above).

The *zeros* and *ones* at the right-hand sides of some table entries are explained in the Key following Table 1. The two entries marked $_{00}$ in Table 2 may be ignored, since it can be demonstrated by the nested method (without the use of series) that the corresponding integrals cancel each other exactly. The three stars (*) in Tables 4 and 6 indicate integrals whose contributions to the divergent parts of I vanish due to symmetric integration. Hence only 48 integrals require evaluation. All necessary momentum-space integrals $J[P]$ are given in Appendix A, leaving only the integrations over the parameters still to be done.

Table 1: Terms of I with 5 denominator factors including $q\cdot n$.

$\not{n}\,(p^2+m^2)$			$2+4\epsilon$ $\;^{0}$	$-2+2\epsilon$ $\;^{0}$	$4-4\epsilon$ $\;^{0}$
$m\,n\cdot k$	$8-8\epsilon$	4	$\substack{0\\0}$	$8-8\epsilon$ $\substack{0\\0}$	$-8+8\epsilon$ $\;^{0}$
$\not{p}\;n\cdot k$	$8-8\epsilon$	-4	$\substack{0\\0}$	$8-8\epsilon$ $\substack{0\\0}$	$-8+8\epsilon$ $\;^{0}$
$n\cdot p\;\not{k}$	$-4+4\epsilon$	$20-8\epsilon$	$\substack{0\\0}$	$-4+4\epsilon$ $\substack{0\\0}$	$4-4\epsilon$ $\;^{0}$
$n\cdot p\;\not{q}$	$4-4\epsilon$ $\;_{1}$	-4 $\;_{1}$	$4+8\epsilon$ $\substack{0\\0}$	$4-4\epsilon$ $\substack{0\\0}$	$-4+4\epsilon$ $\substack{0\\01}$
$\not{n}\;p\cdot q$	$_{1}$	-4 $\;_{1}$	$\substack{0\\0}$	$\substack{0\\0}$	$\substack{0\\01}$
$\not{n}\;k\cdot q$	$_{1}$	$_{1}$	$\substack{0\\0}$	$\substack{0\\0}$	$-4+4\epsilon$ $\substack{0\\01}$
$n\cdot k\;\not{q}$	4ϵ $\;_{1}$	$_{1}$	$4-4\epsilon$ $\substack{0\\0}$	$-8+8\epsilon$ $\substack{0\\0}$	$4-4\epsilon$ $\substack{0\\01}$
$n\cdot k\;\not{k}$	$-4+4\epsilon$	$-4+4\epsilon$ $\;_{0}$	$\substack{0\\00}$	$\substack{0\\00}$	$\substack{0\\00}$
	nq $_\wedge$Kk	n Q$_\wedge$Kk	nqQ$_\wedge$K	nqQ$_\wedge$ k	nqQ Kk

Key to a typical entry:

$\boxed{4 - 4\epsilon \quad {}^{0}_{01}}$ ← value of a at which $\int J_1$ diverges. (See Section 4.)

← values of a at which $\int J_0$ diverges. $_{00}$ indicates that the degree is –1 at $a = 0$. (See Section 4(b).)

coefficient of the term ($\epsilon \equiv 2 - \omega$).

Table 2: Terms of I with 4 denominator factors including $q{\cdot}n$.

$\not n$	$2\epsilon \quad _1$	$-2\epsilon \quad _{01}$	$-2-2\epsilon \quad _0$	$-4-2\epsilon$	$-2+2\epsilon \quad _{00}$	$2-2\epsilon \quad _{00}$
	n $_\wedge$Kk	n Q Kk	n Q$_\wedge$ k	nq $_\wedge$K	nqQ$_\wedge$	nqQ K

Table 3: Covariant-gauge terms of I.

m	$2-2\epsilon+2\epsilon^2 \quad _1$	$_1$	$8 \quad -8\epsilon^2 \quad _0$	$2-2\epsilon+2\epsilon^2 \quad _0$	$-2+2\epsilon-2\epsilon^2 \quad _{01}$
$\not p$	$4-2\epsilon+2\epsilon^2 \quad _1$	$_1$	$8-8\epsilon-8\epsilon^2 \quad _0$	$4-2\epsilon+2\epsilon^2 \quad _0$	$-4+2\epsilon-2\epsilon^2 \quad _{01}$
$\not q$	$4-4\epsilon \quad _1$	$-8+8\epsilon \quad _1$	$-8+4\epsilon+4\epsilon^2 \quad _0$	$-8 \quad +4\epsilon^2 \quad _0$	$4-4\epsilon \quad _{01}$
	q $_\wedge$Kk	Q$_\wedge$Kk	qQ$_\wedge$K	qQ$_\wedge$ k	qQ Kk

Table 4: Terms of I with 5 denominator factors including $q{\cdot}n$ and $k{\cdot}n$.

$\not n \; n{\cdot}p$	$4 \quad {}^0$	$-4 \quad {}^0$	$8 \quad *$	$-4 \quad *$
	nqQ K n	nqQ$_\wedge$ n	n Q$_\wedge$K n	nq $_\wedge$K n

Table 5

$n{\cdot}k \; \not k \, (p^2 + m^2)$	$4 - 4\epsilon \quad {}^0$
	nqQ$_\wedge$Kk

Table 6

$\not n \; n{\cdot}p \; (q{\cdot}k)^2$	$4 \quad *$
	nqQ$_\wedge$Kkn

4. Finding the Divergences

(a) Integration over infinite-parameter space

In order to complete the parameter change from $\{\alpha_j\}$ to $\{A; \lambda, \beta, G, b, h, a\}$, we have to apply the transformation (3.4) to the integrations on the right-hand side of Eq. (3.1) for each of our 48 integrals. Integrals containing all seven "factors" n, q, Q, $\wedge$, K, k, n in their denominators then transform as follows:

$$\begin{aligned} I_E[f] &= \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_7\, J[P] \\ &= \int_0^{\frac{1}{2}} dG \int_G^{1-G} da \int_G^{1-a} d\lambda \int_G^a dh \int_G^\lambda d\beta \int_G^h db \int_0^\infty A^6\, dA\, J[P]\,. \end{aligned} \tag{4.1}$$

But let's suppose now that a certain denominator factor is missing from the simplified integrand f in Eq. (3.1). In that case, as mentioned in Section 3(c), we must set the corresponding α parameter to zero in Eqs. (3.4), and omit integration over this parameter in Eq. (3.1). To see the effect of these changes on the *transformed* integral (4.1), consider the example (3.16), in which $f = \not{q}/(\mathrm{nqQ}_\wedge\mathrm{k})$. Since the fifth and seventh denominator factors are absent in this case, we put $\alpha_5 = \alpha_7 = 0$ to get

$$\begin{aligned} I_E[f] &= \int_E d^{2\omega} q \int_E d^{2\omega} k \frac{\not{q}}{\mathrm{nqQ}_\wedge\ \mathrm{k}} \\ &= \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_4 \int_0^\infty d\alpha_6\, J\left[\frac{-n^*{\cdot}q\ \not{q}}{n_0^2}\right]_{\alpha_5 = \alpha_7 = 0} \\ &= \int_0^{\frac{1}{2}} dG \int_G^{1-G} da \int_G^{1-a} d\lambda \int_G^\lambda d\beta\, \Big[J_A(\lambda, \beta, G, b, h, a) \Big]_{\substack{b \to G \\ h \to a}}\,, \end{aligned} \tag{4.2}$$

$$\text{with} \qquad J_A \equiv \int_0^\infty A^4\, dA\, J\left[\frac{-n^*{\cdot}q\ \not{q}}{n_0^2}\right] = -\int_0^\infty dA\, \frac{A^4}{n_0^2}\, \gamma^\mu\, J\big[n^*{\cdot}q\ q_\mu\big]\,. \tag{4.3}$$

In the next few pages, we shall concentrate on the detailed evaluation of the integral (4.2).

In view of Eqs. (A.4) and (A.8), we find that J_A in Eq. (4.3) becomes

$$J_A \;=\; -\,\frac{\pi^{4-2\epsilon}}{n_0^2}\int_0^\infty dA\,\frac{A^{2\epsilon}\;\mathbf{e}^{-AH}}{D_\parallel D_\perp^{1-\,\epsilon}}\left(n^*{\cdot}s\;\not{s}\;+\;\frac{a\not{n}^*}{2AD_\parallel}\right). \tag{4.4}$$

Integration over A is straightforward. Note that Eq. (4.4) shows the entire A-dependence of the integrand, since $r,\ s,\ D_\parallel,\ D_\perp,$ and H have been defined so as to be independent of A. Using Eqs. (3.14), together with

$$\int_0^\infty t^x\,\mathbf{e}^{-t}\,dt \;=\; \Gamma(x+1)\,, \qquad \mathrm{Re}(x+1)>0\,, \tag{4.5}$$

we find that

$$\Big[\,J_A\,\Big]_{\substack{b\,\to\,G\\ h\,\to\,a}} \;=\; -\,\frac{\pi^{4-2\epsilon}}{n_0^2}\,\big[\,\Gamma(2\epsilon)\;J_0\;+\;\Gamma(1+2\epsilon)\;J_1\;+\;\Gamma(2+2\epsilon)\;J_2\,\big]\,, \tag{4.6}$$

where

$$J_0 = \frac{\not{n}^* a H^{-2\epsilon}}{2D_\parallel^2 D_\perp^{1-\,\epsilon}}\,, \qquad J_1 = \frac{n^*{\cdot}p\,(a\beta-G^2)^2}{D_\parallel^2 D_\perp^{1-\,\epsilon} H}\left(\frac{\not{p}_\parallel}{D_\parallel}+\frac{\not{p}_\perp}{D_\perp}\right) H^{-2\epsilon}\,, \tag{4.7}$$

and $J_2 = 0$. Similar results may be obtained for the other integrals in Tables 1 to 6; only the expressions for J_0, J_1, and J_2 vary.

As explained in the <u>Introduction</u>, the first pole term $\Gamma(2\epsilon)\,J_0$ in Eq. (4.6) arises from integration over the type I parameter A. This divergence may be traced back to the fact that the original integral I is UV-divergent with respect to q and k taken together. Additional UV *subdivergences* will emerge when the J_0 and J_1 terms are integrated over the *finite parameters* $\lambda, \beta, G, b, h,$ and a (called type II parameters in the <u>Introduction</u>). These subdivergences arise from the two loops taken separately.

In view of the appearance of $D_{\|}$, $D_{\perp}$, and H in Eq. (4.7), exact integration of J_0 and J_1 over the finite parameters would pose a major challenge. Fortunately, however, we are only interested in contributions to I which diverge as $\omega \to 2$; i.e., in the terms with negative indices in the Laurent series

$$I(\epsilon) \;=\; I_{-t}(p,n;m)\epsilon^{-t} \;+\; I_{1-t}(p,n;m)\epsilon^{1-t} \;+\; \ldots\,, \tag{4.8}$$

where t is an integer which, in our case, happens to be 2. The divergences occur only at certain boundaries of the region of integration in parameter space. We can extract the divergent terms exactly by integrating the first one or two terms of suitably chosen series which converge to $J[P]$ near these boundaries. In choosing these series, we must be careful to avoid the types of problems which plague the nested method.

In the case of the J_0 term, one boundary at which the parameter integrations diverge as $\omega \to 2$ is just $A = 0$; this fact explains why integration over A generates the divergent factor $\Gamma(2\epsilon)$ in this term. Of course, the presence of this factor implies that the coefficient I_{-1} in Eq. (4.8) will depend on the *finite* part of the integral of J_0 over the remaining parameters. It would seem, therefore, that the J_0 terms will have to be integrated over the *entire* region of integration Φ of these parameters.

Fortunately, things are not quite as bad as they seem, since the most complicated part of the J_0 term is the factor $H^{-2\epsilon}$. If we use the expansion

$$H^{-2\epsilon} \;=\; \mathbf{e}^{-2\epsilon\ln H} \;=\; 1 \;-\; 2\epsilon\ln H \;+\; 2\epsilon^2(\ln H)^2 \;-\; \ldots\;, \tag{4.9}$$

it would appear that we only need to keep the first term of the series when integrating over the whole region Φ, since the common factor of 2ϵ in the other terms will cancel the divergence arising from the factor $\Gamma(2\epsilon)$ in Eq. (4.6). However, it was just this kind of reasoning that led to the catastrophe in the nested method

exemplified by Eq. (2.11). To avoid a similar catastrophe here, we must verify that $\int_\Phi J_0$ is convergent at *all zeros and poles* of H before we may use the series (4.9). It follows from definition (A.9) that H has no poles in Φ, but H does go to *zero* when $b + \beta - G \to 0$. In order to determine whether the integral of any J_0 term diverges at any of these zeros as $\omega \to 2$, and in order to enable us to find all divergent contributions to the final result I, we must systematically determine the location in finite-parameter space of every subdivergence of every J_0, J_1, and J_2 integral. This task will be tackled in the next subsection.

(b) Subdivergences

Divergences in the finite-parameter integrations, as $\omega \to 2$, could potentially occur at the zeros of any of the three factors $D_\parallel$, $D_\perp$, and H which appear in the denominators of the J_0, J_1, and J_2 terms. These zeros occur on certain boundaries of the integration region Φ, such as $a = 0$, $\lambda = h = G$, etc. Specifically,

$$\left.\begin{array}{lll} D_\parallel \to 0 & \text{linearly} & \text{if} \quad a \to 0, \quad a \to 1, \quad \text{or} \quad a, G \to \tfrac{1}{2}\,, \\ D_\perp \to 0 & \left\{ \begin{array}{l} \text{quadratically} \\ \text{or else linearly} \end{array} \right. & \begin{array}{l} \text{if} \quad \textit{both} \quad \lambda,\ h \to 0\,, \\ \text{if} \quad \lambda \to 0, \quad h \to 0, \quad \text{or} \quad \lambda, h \to G\,, \end{array} \end{array}\right\} \tag{4.10a}$$

(cf. Eqs. (3.7) and (4.1)). To locate the zeros of H, we observe that the exponent of the integrand in Eq. (3.3) is less than or equal to zero by construction, and equal to $-AH$ when $\mathbf{y} = 0$ (cf. Eqs. (A.9) and (3.9)). Hence $H \geq 0$ even if the mass m vanishes, while for $m \neq 0$,

$$H \to 0 \quad \text{linearly,} \quad \text{if and only if both} \quad b, \beta \to 0\,. \tag{4.10b}$$

Depending on the specific form of the integrand, a given finite-parameter integral could conceivably diverge at the intersection of some particular subset

of the above boundaries, but not at any other points. Accordingly, we must consider all possible intersections of these boundaries for every term of every finite-parameter integrand. The set of possibilities which needs to be examined is severely constrained by two chains of inequalities, which follow from Eqs. (3.4):

$$a \ \geq \ h \ \geq \ b \ \geq \ G \ \geq 0 \qquad \text{and} \qquad 1-a \ \geq \ \lambda \ \geq \ \beta \ \geq \ G \ \geq 0\,. \tag{4.11}$$

If any particular parameter in either of these chains goes to zero, then all parameters to its right in the chain must also approach zero.

In order to determine whether one of our integrals is divergent at a particular boundary of Φ, we observe that the integral of a rational function of the parameters diverges at some point in finite-parameter space *only if the degree of the integrand* (including the measure $dG\,da\ldots$) *is less than or equal to zero at this point.* The degree of the integrand may be calculated according to the following rules:

1) A linear function of a parameter has degree 1 if it goes to zero at the point in question; otherwise the function has degree 0.
2) The degree of a product (quotient) is the sum (difference) of the degrees of its parts.
3) The degree of a sum or difference, in the numerator, is the *minimum* of the degrees of its parts. For example, if $f \to 0$ linearly and $g \to 0$ quadratically, then $f+g \to 0$ linearly. (If the parts have equal degree, this rule may predict divergences which eventually cancel each other.)
4) The degrees of the denominator factors $D_{\|}$, $D_{\perp}$, and H are given by Eqs. (4.10) for all boundaries except those at which the degrees are zero.
5) The degree of the measure is the dimensionality of finite-parameter space minus the dimensionality of the boundary in question (not counting dimensions associated with absent parameters, such as b and h in Example (4.2)).

For instance, at the boundary $a=0$, inequalities (4.11) imply that G must also be zero; hence the degree of the measure $dG\, da\, d\lambda\, d\beta$ at this boundary is $4-2=2$. To see why, consider a transformation of a and G to polar co-ordinates.

As an example, consider the three terms comprising J_0 and J_1 in Eqs. (4.7). If we multiply these terms by $dG\, da\, d\lambda\, d\beta$ to complete the integrands, we find that the degrees of these integrands are as shown in Tables 7 and 8. Each number in these tables represents the degree of the integrand at the boundary at which the parameters, shown above it and to the left, go to zero. From these tables we see that all three terms in Eqs. (4.7) have subdivergences at the boundary where $a \to 0$ but *not* $\beta \to 0$ or $\lambda \to 0$.

Table 7: Degree of J_0 at various boundaries.

	G	β	λ	$1-a$
G	1	2	2	1
a	0	1	1	

Table 8: Degrees of J_1 terms at various boundaries.

	G	β	λ	$1-a$
G	1	3	3	1
a	0	2	2	

	G	β	λ	$1-a$
G	1	3	2	1
a	0	2	1	

When tables, similar to Tables 7 and 8, are constructed for every term of every finite-parameter integral which arises in the evaluation of I, it is found that subdivergences occur only at the following two boundaries of Φ:

1) $a \to 0$ and *not* $\beta \to 0$ (corresponding to divergent k integration).

2) $a \to 1$ and *not* $b \to 0$ (corresponding to divergent q integration).

Each integral which diverges at one or both of these boundaries is marked in Tables 1 to 6 with ^{0} and/or ^{1}, as explained in the Key. Note that *neither* of these types of divergence occurs at a boundary where b and β *both* go to zero.

In particular, we see from Eq. (4.10b) that there are no subdivergences at zeros of H. Therefore, it is safe to use the series (4.9) in all cases, as we shall do from now on.

To calculate the divergent contributions to I which arise at either of the boundaries $a=0$ or $a=1$, we expand each integrand J_i $(i=0,1,2)$ as a series in powers of the parameters which go to zero at that boundary, and keep only those terms whose integrals diverge. (We justify this procedure by using the five rules given above to show that the integral of the difference between J_i and the terms we keep is convergent at all boundaries in every case.) In all but two of the cases, the degree of the original integrand is zero at the subdivergence, so that only the leading term of the series is needed. For each of the other two cases (marked $_{00}$ in Table 2), the degree of the J_0 integrand is -1, and two terms would be needed. Fortunately, we are saved from having to integrate these terms, since these two exceptional integrals cancel each other exactly (cf. Section 3(c)).

5. Finite-Parameter Integration

(a) Integrations at subdivergences

Employing the two series expansions discussed in Section 4, we can now complete the finite-parameter integrations to the extent necessary to obtain exact expressions for the coefficients I_{-1} and I_{-2} in Eq. (4.8). Proceeding with Example (4.2), we first substitute the series for $H^{-2\epsilon}$ in Eq. (4.9) into Eqs. (4.7). We then use Eq. (4.6) and the identity $\Gamma(z+1)=z\Gamma(z)$ to obtain

$$\Big[J_A\Big]_{\substack{b\,\to\,G\\ h\,\to\,a}} \;=\; -\frac{\pi^{4-2\epsilon}}{n_0^2}\left[\Gamma(2\epsilon)\;Y_0\;+\;\Gamma(1+2\epsilon)\;Y_1\;+\;\Gamma(2+2\epsilon)\;Y_2\right]\;+O(\epsilon), \tag{5.1}$$

$$\text{where} \qquad Y_0 \;=\; \frac{\not{n}^*\, a}{2D_{\|}^2 D_{\perp}^{1-\,\epsilon}}\,, \tag{5.2a}$$

$$Y_1 \;=\; \frac{n^*{\cdot}p\,(a\beta - G^2)^2}{D_{\|}^2 D_{\perp}^{1-\,\epsilon} H}\left(\frac{\not{p}_{\|}}{D_{\|}} + \frac{\not{p}_{\perp}}{D_{\perp}}\right) - \frac{\not{n}^*\, a\, \ln H}{2D_{\|}^2 D_{\perp}^{1-\,\epsilon}}\,, \tag{5.2b}$$

and $Y_2 = 0$. Similar results may be obtained for the other integrals in Tables 1 to 6; only the expressions for Y_0, Y_1, and Y_2 vary. It remains to integrate these expressions over the finite parameters. Since Y_1 and Y_2 are not multiplied by divergent Γ-functions, they contribute to I_{-1} only through the subdivergences in their finite-parameter integrals. The single Y_2 integral has *no* subdivergences, and may be ignored. The $O(\epsilon)$ term in Eq. (5.1) may also be ignored, since there are no subdivergences at the zeros of H, as explained in Section 4(b).

Returning to our example in Eqs. (5.2), we now compute the divergent part of the integral of Y_1 over the finite parameters β, λ, a and G (cf. Eq. (4.2)). We recall from Section 4(b) that all subdivergences of Eq. (4.2) occur at the boundary $a = G = 0$. Therefore, we expand Y_1 as a series in the parameters a and G, and integrate only the divergent leading term. To derive this term, we simply replace H by its leading term $H_0 \equiv H_{a\to 0}$, and drop G^2 and $(1-a)$ from $D_{\|}$, $D_{\perp}$, and from the numerator in Eq. (5.2b). In this way we obtain

$$Y_1 \;=\; a^{\epsilon-2}\, E \;+ O(a^{\epsilon-1}), \tag{5.3}$$

$$\text{where} \qquad E \;\equiv\; \frac{n^*{\cdot}p\,\beta^2}{\lambda^{1-\,\epsilon} H_0}\left(\not{p}_{\|} + \frac{\not{p}_{\perp}}{\lambda}\right) - \frac{\not{n}^*\ln H_0}{2\,\lambda^{1-\,\epsilon}}\,. \tag{5.4}$$

Applying Eqs. (A.9), (3.14), and (3.7), and recalling that $a = h$ in our example, and that $0 \le b \le a$ (cf. inequality (4.11)), we find

$$\left.\begin{aligned} H_0 \;=\; p_{\|}^2\,\beta\left(R - \beta - \frac{\beta S}{\lambda}\right), \qquad &\text{where} \quad R \equiv (p^2 + m^2)/p_{\|}^2\,, \\ &\text{and} \qquad S \equiv p_{\perp}^2/p_{\|}^2\,. \end{aligned}\right\} \tag{5.5}$$

Integration of Eq. (5.3) over β, λ, a, and G in accordance with Eq. (4.2) yields

$$\begin{aligned}\int_\Phi Y_1 \; &= \int_0^{\frac{1}{2}} dG \int_G^{1-G} da \int_G^{1-a} d\lambda \int_G^{\lambda} d\beta \; a^{\epsilon-2} \, E \quad + \text{ finite terms}\,, \\ &= \int_0^{\frac{1}{2}} dG \int_G^{1-G} a^{\epsilon-2} \, da \; (L_0 - L_1) \quad + \text{ finite terms}\,, \qquad (5.6)\end{aligned}$$

where
$$L_0 \;\equiv\; \int_0^1 d\lambda \int_0^{\lambda} d\beta \; E\,, \qquad (5.7a)$$

$$L_1 \;\equiv\; \int_{1-a}^1 d\lambda \int_G^{\lambda} d\beta \; E \; + \int_0^G d\beta \int_\beta^1 d\lambda \; E\,, \qquad (5.7b)$$

and E is defined by Eq. (5.4). Integral (5.7a) is convergent even for $\omega = 2$ ($\epsilon = 0$), and may be evaluated in the $\epsilon = 0$ case with the help of formulas (B.9), (B.8), and (B.5) from Appendix B. The two integration regions in Eq. (5.7b) are entirely inside the region in Eq. (5.7a), and shrink in proportion to a. Consequently, $L_1 \to 0$ as $a \to 0$; hence the integral of L_1 over a and G may be absorbed into the finite terms in Eq. (5.6). L_0 may then be factored out of the integral, since it is independent of a and G. The a and G integrations are easily completed: they produce the divergent factor ϵ^{-1}, plus a finite term, so that Eq. (5.6) becomes

$$\int_\Phi Y_1 \;=\; \frac{1}{\epsilon} \int_0^1 d\lambda \int_0^{\lambda} d\beta \; E_{\epsilon=0} \quad + \text{ finite terms}\,. \qquad (5.8)$$

The Y_1 terms arising from the other integrals in Tables 1 to 6 may be integrated in similar fashion. After convergent terms have been discarded, each integral factors into a trivial divergent integral, times a finite integral which can be evaluated at $\epsilon = 0$ with the help of the formulas from Appendix B. Adding the results of *all* Y_1 integrations, we finally obtain the total contribution to the *divergent part* of I from all Y_1 terms (in Minkowski space):

$$
\begin{aligned}
I_{Y_1} \;=\; \frac{\Gamma(4-2\omega)}{(4\pi)^{2\omega}} \Bigg[& 24\,\not{n}\left(\frac{m^2-p^2}{n\cdot p}\right)\left[L_2\left(\frac{p^2}{p^2-m^2}\right) - L_2\left(\frac{p^2+B}{p^2-m^2}\right)\right] + \\
& \mathbf{N}\cdot(\,8,\ \ 22, -39,\ \ 89,\ \ 0\,) + \\
& \left[\frac{m^2}{p^2} + \left(\frac{m^4}{p^4}-1\right)\ln\left(1-\frac{p^2}{m^2}\right) - \ln(m^2)\right]\ \mathbf{N}\cdot(\,0,\ -8,\ \ 4,\ -4,\ \ 0\,) + \\
& \left[\left(\frac{m^2}{p^2}-1\right)\ln\left(1-\frac{p^2}{m^2}\right) - \ln(m^2)\right]\ \mathbf{N}\cdot(\,8,\ \ 0,\ \ 0,\ \ 0,\ 24\,) + \\
& \left[\left(\frac{m^2-p^2}{B+p^2}\right)\ln\left(\frac{B+m^2}{m^2-p^2}\right) + \ln(B+m^2)\right]\ \mathbf{N}\cdot(\,0, -16,\ \ 16, -32,\ 24\,)\Bigg],
\end{aligned}
\tag{5.9}
$$

where $\qquad B \equiv -\dfrac{n\cdot p\, n^*\cdot p}{n_0^2}, \qquad L_2(z) \equiv \displaystyle\int_0^1 \frac{\ln x\, dx}{x - 1/z}\ ;$

the vector $\mathbf{N}$ is given by:

$$
\mathbf{N} \equiv \left(m,\ \not{p},\ \frac{n\cdot p\,\not{n}^*}{n\cdot n^*},\ \frac{n^*\cdot p\,\not{n}}{n\cdot n^*},\ \frac{p^2\,\not{n}}{n\cdot p}\right). \tag{5.10}
$$

(b) Integrations over all parameter-space

Finally, we must integrate the Y_0 terms over the finite parameters λ, β, G, b, h, and a (cf. Eqs. (5.1) and (4.2)). Excluding the $\Gamma(2\epsilon)$ factor produced by the A integration, the integral of a Y_0 term will generally include a *divergent* term proportional to ϵ^{-1}, plus a *finite* term which is independent of ϵ, plus terms which go to zero as $\epsilon \to 0$. Due to the $\Gamma(2\epsilon)$ factor, the *divergent* term will contribute to I_{-2} (cf. Eq. (4.8)), while the *finite* term will contribute to I_{-1}. Since we need both I_{-2} and I_{-1}, we must evaluate both the divergent and finite parts of the integral of Y_0. The divergent part comes from integration of Y_0 at its subdivergences, while the finite part comes from integration over the whole

region Φ. Since the subdivergences are part of the integral over the whole region, we can extract both the divergent part *and* the finite part by integrating Y_0 over all of Φ.

The functional form of the Y_0 terms is simpler than that of the Y_1 terms, because the former involve only the first term of the series in Eq. (4.9) and are, therefore, independent of H. However, when the denominator factor $\wedge$ is present in the original integrand $f(q,k)$ (i.e., when $G \neq 0$), integration of the Y_0 terms is still complicated by the presence of the factors $D_\| = a(1-a) - G^2$ and $D_\perp = \lambda h - G^2$ (see, for instance, Eq. (5.2a)). In the integration of the Y_1 terms, we simplified these factors by dropping all terms except those of lowest degree in $\{a, G\}$ or $\{1-a, G\}$. This procedure is, unfortunately, of no use here, because higher-order terms contribute to I_{-1} via the *finite* part of the integral of Y_0. Our new strategy is, therefore, as follows: we first simplify the $D_\|$ and $D_\perp$ factors by changing the integration variables from $\{G, a, \lambda, \beta, b\}$ to $\{V, U, X, \tau, t\}$, according to the following definitions:

$$G = Va\,, \quad a = \frac{1}{U + V^2 + 1}\,, \quad \lambda = a(X + V^2)\,, \quad \beta = \tau a\,, \quad b = ta\,. \tag{5.11}$$

Note that this change of variables does not involve the parameter h, and is, therefore, inapplicable to any integral containing the factor $n\cdot k$ in the denominator of the original integrand $f(q,k)$. This restriction is not as severe as it might seem, because we may always interchange q and k before integrating, as mentioned in Section 3(c). Hence the transformation (5.11) may be applied to all integrals except those with *both* $n\cdot k$ and $n\cdot q$ in the denominator of f. Fortunately, all such integrals of interest here (see Table 4) are harmless anyway, since they give rise to Y_0 terms which are proportional to $n^* n^*$, and hence vanish in the light-cone gauge.

Under the change of variables (5.11), the boundaries $a = 0$ and $a = 1$ (at which all subdivergences occur) are transformed to $U \to \infty$ and $U = V = 0$, respectively, while the integrations in Eq. (4.1) are transformed according to:

$$
\begin{array}{ccccc}
\displaystyle\int_0^{\frac{1}{2}} dG & \displaystyle\int_G^{1-G} da & \displaystyle\int_G^{1-\,a} d\lambda & \displaystyle\int_G^{\lambda} d\beta & \displaystyle\int_G^{a} db\,, \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\displaystyle\int_0^{1} dV & \displaystyle\int_{V-V^2}^{\infty} dU\; a^d & \displaystyle\int_{V-V^2}^{U} dX & \displaystyle\int_V^{X+V^2} d\tau & \displaystyle\int_V^{1} dt\,,
\end{array}
\tag{5.12}
$$

where d is the number of factors in the denominator of the original integrand f, a^d is the Jacobian determinant of the transformation, and parameter integrations corresponding to absent denominator factors are to be omitted, as already explained in Section 4(a).

Returning to our example, we now integrate Eq. (5.2a) over the new parameters. From Eqs. (5.11) and (3.7), we deduce $D_\parallel = a^2 U$ and $D_\perp = a^2 X$, and since $\int db$ does not appear in Eq. (4.2), $\int dt$ is absent from the transformed integral:

$$
\begin{aligned}
\int_\Phi Y_0 &= \int_\Phi \frac{\not{n}^*\, a}{2 D_\parallel^2 D_\perp^{1-\,\epsilon}} = \int_0^1 dV \int_{V-V^2}^{\infty} dU\; a^5 \int_{V-V^2}^{U} dX \int_V^{X+V^2} d\tau\; \frac{\not{n}^*\, a^{-5+2\epsilon}}{2\, U^2 X^{1-\,\epsilon}}\,, \\
&= \frac{\not{n}^*}{2} \int_0^1 dV \int_{V-V^2}^{\infty} dU\; a^{2\epsilon} \left[\frac{U^{\epsilon-1}}{1+\epsilon} + \frac{(V-V^2)^{1+\,\epsilon}}{(1+\epsilon)\,\epsilon\, U^2} - \frac{V-V^2}{\epsilon\, U^{2-\,\epsilon}} \right].
\end{aligned}
\tag{5.13}
$$

Since the old parameter a is equal to the complicated function $(U + V^2 + 1)^{-1}$, the new expressions $a^2 U$ and $a^2 X$ for $D_\parallel$ and $D_\perp$ may not seem to represent a significant improvement over the old ones. However, when all factors of a in the transformed integrand, including the Jacobian a^d, are collected, it turns out that the net power of a is equal to 2ϵ in every case. This fact suggests that we may be able to obtain the divergent and finite parts of the Y_0 integrals by writing $(U + V^2 + 1)^{-2\epsilon}$ in series form, and keeping only the first one or two terms.

In order to avoid the catastrophe of Eq. (2.11), we need series which converge uniformly over the whole domain of the U, V integrations. For $U > 2$, the binomial series

$$(U + V^2 + 1)^{-2\epsilon} \;=\; U^{-2\epsilon}\left[1 \;-\; \frac{2\epsilon}{1}\frac{(V^2+1)}{U} \;+\; \frac{2\epsilon}{1}\frac{(2\epsilon+1)}{2}\frac{(V^2+1)^2}{U^2} \;-\; \dots\right] \tag{5.14a}$$

is suitable, since its convergence accelerates with increasing U. On the other hand for $U \le 2$, an exponential series is appropriate:

$$(U + V^2 + 1)^{-2\epsilon} \;=\; 1 \;-\; 2\epsilon\,\ln(U + V^2 + 1) \;+\; 2\epsilon^2\,(\ln(U + V^2 + 1))^2 \;-\; \dots\;. \tag{5.14b}$$

In practice, one may avoid splitting up the integration region by using Eq. (5.14a) for integrals which diverge as $U \to \infty$ (e.g.: the first term in brackets in Eq. (5.13)), and Eq. (5.14b) for integrals which either diverge for finite U, or which do not diverge at all (e.g.: the other two terms). This strategy is valid, because in those parts of the integration region where an integral converges, the finite part of the integral may always be found by setting $\epsilon = 0$, either before or after integration. If we set $\epsilon = 0$ in Eq. (5.14a) or (5.14b), all terms except for the first one go to zero, and the two series become identical.

It turns out that, even at the points in the region of integration where a Y_0 integral diverges, only the *first term* of the appropriate series contributes to the divergent and finite parts of the integral. To see why, we recall from Section 4(b) that the degree of the integrand of every divergent finite-parameter integral is zero at the point in parameter-space where the divergence occurs; in other words, every integral diverges like $\int dx/x$ at $x = 0$; if the degree were greater even by one, the integral would not diverge. In the series (5.14a), all terms but the first are of higher degree in $1/U$ than the factor $(U + V^2 + 1)^{-2\epsilon}$, so that the

integrals involving these terms will remain finite as $U \to \infty$. The accompanying factors of ϵ in Eq. (5.14a) ensure that these terms do not even contribute to the *finite* part of the Y_0 integral as $\epsilon \to 0$. Hence we may drop all terms of the series, except the first one. A similar argument applies to the series (5.14b), where $\ln(U+V^2+1) \to 0$ linearly as $U, V \to 0$.

The preceding analysis tells us that the complicated factor $(U+V^2+1)^{-2\epsilon}$, appearing in the transformed integrands of the Y_0 integrals, may always be replaced, either by $U^{-2\epsilon}$ or by 1. Once this replacement has been made, integration of the Y_0 terms over t, τ, X, and U is trivial, while final integration over V is easily accomplished with the help of the formula

$$\int_0^1 V^C(1-V)^D\,dV \;=\; \frac{\Gamma(C+1)\,\Gamma(D+1)}{\Gamma(C+D+2)}\,; \tag{5.15}$$

here C and D are linear functions of ϵ.

The results of the integration of all Y_0 terms are shown in Appendix C. When these results are multiplied by the appropriate coefficients and then added, the total contribution to the divergent part of I from all Y_0 terms reads as follows (in Minkowski space):

$$I_{Y_0} \;=\; \frac{\Gamma(4-2\omega)}{(4\pi)^{2\omega}}\,\mathbf{N}\cdot\left(\frac{(\,4,\;4,-6,\;14,\;0\,)}{2-\omega} \;+\; (\,8,-4,\;11,\;3,\;0\,)\right), \tag{5.16}$$

where $\mathbf{N}$ has already been defined in Eq. (5.10). To obtain the grand total of all divergent parts of I, we simply add I_{Y_0} from Eq. (5.16) to I_{Y_1} from Eq. (5.9). Thus (cf. Eq. (2.7))

$$\begin{aligned} I_0 \;&=\; ig^4\, T^b_{DA}T^a_{CD}T^b_{BC}T^a_{AB}\, I \\ &=\; ig^4\, T^b_{DA}T^a_{CD}T^b_{BC}T^a_{AB}\,[I_{Y_0}+I_{Y_1}\;+\text{finite terms}]. \end{aligned} \tag{5.17}$$

6. Concluding Remarks

In this paper we have developed a powerful new integration technique for multi-loop integrals, called the *matrix method*, and used it to compute the divergent part of the overlapping two-loop fermion self-energy function $i\Sigma$ in the light-cone gauge. The overlapping self-energy diagram (Fig. 1) is the most challenging of the two-loop diagrams, especially in a noncovariant gauge such as the light-cone gauge. (The contributions from Figs. 2(a) to 2(d) are calculated in the sequel to this paper.) For completeness, we remind the reader of the one-loop result (Fig. 3) in the light-cone gauge [15]:

$$i\Sigma_{\text{1-loop}} \;=\; \frac{g^2}{12\pi^2(2-\omega)}\left(\not{p} + 2m + 2\,\frac{n^*{\cdot}p\,\not{n} - n{\cdot}p\,\not{n}^*}{n{\cdot}n^*}\right) + \text{finite terms.} \tag{6.1}$$

The main results for the overlapping self-energy function (cf. Fig. 1) may be summarized thus:

(i) The total divergent contribution, given by the sum of Eqs. (5.9) and (5.16), contains both simple and double poles. The double-pole term from Eq. (5.16) reads

$$\frac{\Gamma(4-2\omega)}{(4\pi)^{2\omega}(2-\omega)}\left(4m + 4\not{p} - 6\,\frac{n{\cdot}p\,\not{n}^*}{n{\cdot}n^*} + 14\,\frac{n^*{\cdot}p\,\not{n}}{n{\cdot}n^*}\right), \tag{6.2}$$

which is seen to be *local*, even off mass-shell. The contributions to $[i\Sigma]_{p^2=m^2}$ from results (6.1) and (6.2) are, therefore, strictly local. The coefficient of the single pole, on the other hand, has also non-local terms, as seen from Eq. (5.10). (Both types of poles are local on mass-shell.) The locality of the double-pole term (6.2) strongly suggests that the complete fermion-mass counterterm will likewise be local, but confirmation will depend on the results from Figures 2(a) and 2(b).

(ii) The overlapping fermion self-energy integral contains a maximum of *seven* propagators. Detailed analysis of the singularity structure of the corresponding seven-parameter integrals in the "user-friendly" set $S = \{A; \lambda, \beta, G, b, h, a\}$, Eq. (3.4), reveals that the *first* simple pole originates from integration over the *infinite* parameter A, while the *second* simple pole (and double pole overall!) arises from integration over some of the *finite* parameters λ, β, G, b, h, and a (subdivergences).

(iii) The matrix method is amazingly powerful, being applicable not only to massive and massless integrals, but also to covariant integrals, as depicted in Table 3, and to noncovariant integrals, i.e., those containing the factors $(k{\cdot}n)^{-1}$ and/or $(q{\cdot}n)^{-1}$ (Tables 1, 2, 4, 5, and 6). The success of the technique derives, quite simply, from combining the 2ω-dimensional momentum vectors q_μ and k_μ, and then integrating over 4ω-dimensional Euclidean space in a *single* operation. It is this compact procedure which yields exact formulas (at an intermediate stage), whose analytic structure simplifies the ensuing parameter integrations tremendously. We note in passing that the matrix method works equally well for axial-type gauges, notably the temporal gauge ($n^2 > 0$) and the pure axial gauge ($n^2 < 0$).

(iv) Although it has not been possible to corroborate our final result against other calculations (there just aren't any!), we have nevertheless been able to check all the covariant integrals listed in Table 11. And these agree exactly with the divergent parts of the double covariant integrals currently used in computing radiative corrections in the Standard Model [10,11]. Accordingly we feel reasonably confident that our final answer for the overlapping integral I in Eq. (2.7) is correct.

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Figure Captions

Fig. 1 Two-loop overlapping quark self-energy function (solid lines denote quarks, wavy lines gluons).

Fig. 2 Other two-loop diagrams for the quark self-energy.

Fig. 3 One-loop quark self-energy function computed in ref. [15].

Appendix A: Useful momentum-space integrals.

The following formulas were derived from Eqs. (3.5) to (3.14). Note that $\mathbf{z} \equiv (k_4,\, q_4,\, k_3,\, q_3,\, \dots)^\top$.

$$J[k_\mu] \;=\; r_\mu\, J[1], \tag{A.1}$$

$$J[q_\mu] \;=\; s_\mu\, J[1], \tag{A.2}$$

$$J[n^*{\cdot}q\, k_\mu] \;=\; \left(n^*{\cdot}s\, r_\mu - \frac{G n^*_\mu}{2AD_\parallel}\right) J[1], \tag{A.3}$$

$$J[n^*{\cdot}q\, q_\mu] \;=\; \left(n^*{\cdot}s\, s_\mu + \frac{a n^*_\mu}{2AD_\parallel}\right) J[1], \tag{A.4}$$

$$J[n^*{\cdot}q\, n{\cdot}k\, \not{k}] \;=\; \left(n^*{\cdot}s\, n{\cdot}r\, \not{r} \;+\; \frac{\beta\, n^*{\cdot}p\, \not{n} \;+\; 2Gn_0^2(\not{r} + \not{r}_\parallel)}{2AD_\parallel}\right) J[1], \tag{A.5}$$

$$J[n^*{\cdot}q\, n{\cdot}k\, \not{q}] \;=\; \left(n^*{\cdot}s\, n{\cdot}r\, \not{s} \;+\; \frac{b\, n{\cdot}p\, \not{n}^* \;+\; 2Gn_0^2(\not{s} + \not{s}_\parallel)}{2AD_\parallel}\right) J[1], \tag{A.6}$$

$$J[n^*{\cdot}q\, q{\cdot}k] \;=\; \left(n^*{\cdot}s\, s{\cdot}r \;+\; \frac{b\, n^*{\cdot}p \;+\; O(G)}{2AD_\parallel}\right) J[1], \tag{A.7}$$

$$\text{where} \qquad J[1] \;=\; \left(\frac{\pi}{A}\right)^{4-2\epsilon} \frac{\mathbf{e}^{-AH}}{D_\parallel D_\perp^{1-\,\epsilon}}\,, \qquad (\epsilon \,\equiv\, 2-\omega), \tag{A.8}$$

$$H \;\equiv\; \frac{C - \mathbf{B}{\cdot}\mathbf{m}}{A} \;=\; (b+\beta-G)(p^2+m^2) - (br+\beta s){\cdot}p\,. \tag{A.9}$$

Since J is a linear functional, we can use $J[q_\mu]$ to deduce integrals such as $J[n^*{\cdot}q]$ and $J[\not{q}]$, etc.

Appendix B: Formulas for Y_1 integration

Integration of Y_1 terms requires integration over finite parameters of expressions involving $H_0 \equiv H_{a\to 0}$ and $H_1 \equiv H_{a,\, h\to 1}$. H_0 is given by Eqs. (5.5), which in turn were derived from Eq. (A.9). We may similarly derive $H_1 = p_\parallel^2\, b\,(R - b - bS)$, where R and S are the same as in Eqs. (5.5).

Formulas (B.1) to (B.11) may be obtained by elementary means. Since $p^2 = p_\parallel^2 + p_\perp^2$, it follows from Eqs. (5.5) that $p_\parallel^2\,(R - S - 1) = m^2$.

$$\int_0^1 d\lambda \left[\ln H_0\right]_{\beta=\lambda} = \ln(m^2) - 2 + (R{-}S)\ln\left[\frac{R-S}{R{-}S{-}1}\right], \tag{B.1}$$

$$\int_0^1 db\ \ln H_1 = \ln(m^2) - 2 + \frac{R}{S{+}1}\ln\left[\frac{R}{R{-}S{-}1}\right], \tag{B.2}$$

$$2\int_0^1 db\ b\ln H_1 = \ln(m^2) - 1 + \frac{R^2}{(S{+}1)^2}\ln\left[\frac{R}{R{-}S{-}1}\right] - \frac{R}{S{+}1}, \tag{B.3}$$

$$\int_0^1 d\lambda \int_0^\lambda d\beta\ \frac{\beta\, p_\parallel^2}{\lambda H_0} = L_2\left(\frac{S+1}{R}\right) - L_2\left(\frac{S}{R}\right), \tag{B.4}$$

$$\text{where} \qquad L_2(z) \equiv \int_0^1 \frac{\ln x\ dx}{x - 1/z}\,.$$

Formula (B.4) was derived with the help of the substitution $\beta = y\lambda$, followed by integration over λ and then integration by parts over y. The same substitution is helpful in the derivation of some of the integrals (B.5) to (B.11) in Table 9.

In formulas (B.5) to (B.11), $\mathbf{W} \equiv$

$$\left(\ln(m^2),\ 1,\ \ln\left[\frac{R}{R{-}S{-}\,1}\right],\ (R{-}S)\ln\left[\frac{R-S}{R{-}S{-}\,1}\right],\ L_2\left[\frac{S{+}1}{R}\right] - L_2\left[\frac{S}{R}\right]\right), \tag{B.12}$$

so that, for example, the right-hand side of formula (B.5) is

$$\ln(m^2) \;-\; 4 \;+\; (R-S)\ln\left[\frac{R-S}{R-S-1}\right] \;+\; R\,L_2\left[\frac{S+1}{R}\right] \;-\; R\,L_2\left[\frac{S}{R}\right].$$

Table 9

E	$\int_0^1 d\lambda \int_0^\lambda d\beta \;\; E$	
$\frac{\ln H_0}{\lambda}$	$\mathbf{W}\cdot\left(1, \quad -4 \quad , \quad 0 \quad , \quad 1 \quad , \; R \right)$	(B.5)
$2\,\frac{\beta\,\ln H_0}{\lambda^2}$	$\mathbf{W}\cdot\left(1, \quad -3 \quad , \quad \frac{R^2}{S(S+1)} \quad , \; 1-\frac{R}{S} \; , \; 0 \right)$	(B.6)
$4\,\frac{\beta\,\ln H_0}{\lambda}$	$\mathbf{W}\cdot\left(1, \quad S-3R-2 \quad , \quad \frac{-2\,R^2}{S+1} \quad , \; 3R-S \; , \; 2R^2 \right)$	(B.7)
$\frac{\beta^2\,p_\parallel^2}{\lambda^2 H_0}$	$\mathbf{W}\cdot\left(0, \quad 0 \quad , \quad \frac{R}{S(S+1)} \quad , \; -\frac{1}{S} \; , \; 0 \right)$	(B.8)
$\frac{\beta^2\,p_\parallel^2}{\lambda H_0}$	$\mathbf{W}\cdot\left(0, \quad -1 \quad , \quad \frac{-R}{S+1} \quad , \; 1 \; , \; R \right)$	(B.9)
$2\,\frac{\beta^3\,p_\parallel^2}{\lambda^2 H_0}$	$\mathbf{W}\cdot\left(0, \quad \frac{R}{S+1}-1 \quad , \quad \frac{R^2}{S(S+1)^2} \quad , \; 1-\frac{R}{S} \; , \; 0 \right)$	(B.10)
$2\,\frac{\beta^3\,p_\parallel^2}{\lambda H_0}$	$\mathbf{W}\cdot\left(0, \; \frac{R}{S+1}+S-3R-\frac{1}{2} \; , \; \frac{-R^2(2S+3)}{(S+1)^2} \; , \; 3R-S \; , \; 2R^2 \right)$	(B.11)

Appendix C: Results of Y_0 integration

Each entry in the following tables, when multiplied by $\pi^{2\omega}\Gamma(4-2\omega)$ times the Euclidean-vector expression at the right-hand side of its row of the table, gives the Y_0 portion of the divergent part of the integral of the expression at the left-hand side divided by the denominator at the bottom. Denominators are represented in the notation of definitions (3.15), and $\epsilon \equiv 2-\omega$.

Table 10

k_μ	-1	-1	$-\frac{1}{2\epsilon}+\frac{1}{2}$	$-\frac{1}{2\epsilon}+\frac{1}{2}$	0	$n^*_\mu/n\cdot n^*$
q_μ	$\frac{1}{\epsilon}+1$	$\frac{1}{\epsilon}+1$	$\frac{1}{\epsilon}-1$	$\frac{1}{\epsilon}-1$	$\frac{2}{\epsilon}-2$	$n^*_\mu/n\cdot n^*$
$\not{n}\, k\cdot q$					$\frac{1}{\epsilon}-1$	$\not{n}\, n^*\cdot p/n\cdot n^*$
	$\frac{1}{2\epsilon}+1$	$\frac{1}{2\epsilon}+1$	$\frac{1}{\epsilon}-1$	$\frac{1}{2\epsilon}-\frac{1}{2}$	$\frac{1}{\epsilon}-1$	$\not{n}^*\, n\cdot p/n\cdot n^*$
$n\cdot k\, \not{q}$	2	6	$\frac{2}{\epsilon}-1$	$\frac{2}{\epsilon}+1$	0	$-\not{p}_{\parallel}/8$
	2	6	$\frac{2}{\epsilon}-3$	$\frac{2}{\epsilon}-1$	0	$-\not{p}_{\perp}/8$
	1	$\frac{1}{2\epsilon}+\frac{1}{2}$				$\not{n}\, n^*\cdot p/n\cdot n^*$
$n\cdot k\, \not{k}$	11	9				$-\not{p}_{\parallel}/8$
	5	3				$-\not{p}_{\perp}/8$
	nq ∧Kk	n Q∧Kk	nqQ∧K	nqQ∧ k	nqQ Kk	

Table 11

1	$\frac{1}{\epsilon}+1$	$\frac{1}{\epsilon}+1$	$\frac{1}{\epsilon}+1$	$\frac{1}{\epsilon}+1$	$\frac{2}{\epsilon}$	1
$\not{q}$	$\frac{2}{\epsilon}+1$	$\frac{6}{\epsilon}+7$	$\frac{4}{\epsilon}+2$	$\frac{4}{\epsilon}+6$	$\frac{8}{\epsilon}$	$\not{p}/8$
	q ∧Kk	Q∧Kk	qQ∧K	qQ∧ k	qQ Kk	

Table 12

$\not{n}$	$1/\epsilon$	0	$2/\epsilon$	$(4/\epsilon)-4$	$\not{n}\, n^*\cdot p/n\cdot n^*$
	n ∧Kk	nq ∧K	n Q∧ k	n Q Kk	

Appendix D: Computational summary of a typical overlapping integral.

We summarize the main steps in the computation of the double integral, corresponding to the fifth row and second-last column of Table 1, Section 3(c):

$$
I_E[f] \;\equiv\; \int_E d^{2\omega}q \int_E d^{2\omega}k\; f\,, \qquad \text{with} \quad f = \frac{\not{q}}{\mathrm{nqQ}_{\wedge}\ \mathrm{k}}\,. \tag{D.1}
$$

Using definitions (3.15), we find that

$$
I_E[f] \;=\; \int_E \int_E \frac{\not{q}\;\, d^{2\omega}q\; d^{2\omega}k}{n{\cdot}q\; q^2\,[(p-q)^2+m^2]\;[(p-k-q)^2+m^2]\;k^2}\,. \tag{D.2}
$$

1. First, we perform the exponential parametrization (2.9) with Schwinger parameters $\alpha_1, \ldots, \alpha_7$ $(\alpha_5 = \alpha_7 = 0)$; we then replace these parameters by the "user-friendly" set $\{A; \lambda, \beta, G, b, h, a\}$, defined in Eqs. (3.4). Thus (cf. Eq. (4.2)),

$$
I_E[f] \;= \int_0^\infty d\alpha_1 \;\ldots \int_0^\infty d\alpha_4 \;\int_0^\infty d\alpha_6\; J\left[\frac{-n^*{\cdot}q\;\not{q}}{n_0^2}\right]_{\substack{\alpha_5\,\to\,0\\ \alpha_7\,\to\,0}}, \tag{D.3}
$$

$$
= \int_\Phi \int_0^\infty A^4\, dA\; J\left[\frac{-n^*{\cdot}q\;\not{q}}{n_0^2}\right]_{\substack{b\,\to\,G\\ h\,\to\,a}}, \tag{D.4}
$$

with J defined by Eq. (3.3), and $\displaystyle\int_\Phi \equiv \int_0^{\frac{1}{2}} dG \int_G^{1-G} da \int_G^{1-a} d\lambda \int_G^{\lambda} d\beta\,.$

2. Next, we perform the 4ω-dimensional *momentum integration* in J with the help of formulas (A.4) and (A.8). In this way we obtain (cf. Eq. (4.4)):

$$
I_E[f] \;=\; -\,\frac{\pi^{4-2\epsilon}}{n_0^2}\int_\Phi \int_0^\infty dA\left[\frac{A^{2\epsilon}\;\mathbf{e}^{-AH}}{D_\parallel D_\perp^{1-\,\epsilon}}\left(n^*{\cdot}s\;\not{s} \;+\; \frac{a\not{n}^*}{2AD_\parallel}\right)\right], \tag{D.5}
$$

with $D_\parallel$, $D_\perp$, r, s and H being defined in Eqs. (3.7), (3.14) and (A.9). (In Eq. (D.5), and all subsequent equations, $b \to G$, $h \to a$, and $\epsilon \equiv 2-\omega$.)

3. Integration over the infinite (type I) parameter A yields the *first* simple pole (cf. Eq. (4.6)):

$$I_E[f] \;=\; -\,\frac{\pi^{4-2\epsilon}}{n_0^2}\int_\Phi \big[\,\Gamma(2\epsilon)\;J_0 \;+\; \Gamma(1+2\epsilon)\;J_1\,\big], \tag{D.6}$$

with J_0 and J_1 given in Eqs. (4.7). The term with $\Gamma(2\epsilon)$ diverges as $\epsilon \to 0$ ($\omega \to 2$). Additional *subdivergences* will arise from integration over Φ.

4. Both J_0 and J_1 include factors of $H^{-2\epsilon}$. According to Section 4(b), there are no subdivergences at the zeros of H; therefore, we use the exponential series for $H^{-2\epsilon}$ given in Eq. (4.9) to get (cf. Eq. (5.1)):

$$I_E[f] \;=\; -\,\frac{\pi^{4-2\epsilon}}{n_0^2}\int_\Phi \big[\,\Gamma(2\epsilon)\;Y_0 \;+\; \Gamma(1+2\epsilon)\;Y_1 \;\; + O(\epsilon)\,\big]\,, \tag{D.7}$$

with Y_0 and Y_1 shown in Eqs. (5.2). As discussed in Section 5, we require both the *finite* and *divergent* parts of $\int_\Phi Y_0$, but only the divergent part of $\int_\Phi Y_1$.

5(a). In this particular example, all subdivergences occur at $a = G = 0$. (In general, according to Section 4(b), subdivergences may also occur at $a = 1$, $G = 0$.) To find the divergent part of $\int_\Phi Y_1$, therefore, we integrate only the portion of Y_1 of least degree in $\{a, G\}$ (cf. Eqs. (5.3) to (5.7)):

$$\int_\Phi Y_1 \;=\; \int_0^{\frac{1}{2}} dG \int_G^{1-G} da \int_G^{1-a} d\lambda \int_G^{\lambda} d\beta\;[\,a^{\epsilon-2}\,E \;+O(a^{\epsilon-1})\,]\,, \tag{D.8}$$

$$=\; \int_0^{\frac{1}{2}} dG \int_G^{1-G} a^{\epsilon-2}\,da \;\int_0^{1} d\lambda \int_0^{\lambda} d\beta\;E \quad+\;\text{finite terms}\,, \tag{D.9}$$

with E being defined in Eq. (5.4). Since E is independent of a and G,

$$\int_\Phi Y_1 \;=\; \frac{1}{\epsilon}\int_0^1 d\lambda \int_0^{\lambda} d\beta\;E_{\epsilon=0} \;\;+\;\text{finite terms}\,. \tag{D.10}$$

5(b). Integration over λ and β with the help of Eqs. (B.9), (B.8), and (B.5) yields:

$$\int_\Phi Y_1 \;=\; \frac{\mathbf{W}}{\epsilon}\cdot\left[\frac{n^*\!\cdot p\, \not{p}_\parallel}{p_\parallel^2}\left(0,\,-1,\,\frac{-R}{S+1},\,1,\,R\right) \;-\; \frac{\not{n}^*}{2}(1,\,-4,\,0,\,1,\,R)\right.$$
$$\left. +\; \frac{n^*\!\cdot p\, \not{p}_\perp}{p_\parallel^2}\left(0,\,0,\,\frac{R}{S(S+1)},\,\frac{-1}{S},\,0\right)\right] \;+\;\text{finite terms}\,, \qquad \text{(D.11)}$$

with $\mathbf{W}$ being defined in Eq. (B.12), and R and S in Eqs. (5.5).

6(a). To integrate Y_0 over Φ, we first change the integration variables from $\{a,\,G,\,\lambda,\,\beta\}$ to $\{U,\,V,\,X,\,\tau\}$ via Eqs. (5.11). From Eqs. (3.7) we deduce $D_\parallel = a^2U$ and $D_\perp = a^2X$, so that the integral of Eq. (5.2a) over Φ becomes (cf. Eq. (5.13)):

$$\int_\Phi Y_0 \;=\; \int_0^1\! dV \int_{V-V^2}^{\infty}\! dU\; a^5 \int_{V-V^2}^{U}\! dX \int_V^{X+V^2}\! d\tau\; \frac{\not{n}^*\, a^{-5+2\epsilon}}{2\,U^2X^{1-\,\epsilon}}\,,$$
$$=\frac{\not{n}^*}{2}\int_0^1\! dV \int_{V-V^2}^{\infty}\! dU\; a^{2\epsilon}\left[\frac{U^{\epsilon-1}}{1+\epsilon} + \frac{(V-V^2)^{1+\,\epsilon}}{(1+\epsilon)\,\epsilon\, U^2} - \frac{V-V^2}{\epsilon\, U^{2-\,\epsilon}}\right], \qquad \text{(D.12)}$$

with $a = (U+V^2+1)^{-1}$, according to Eqs. (5.11).

6(b). As explained in Section 5(b), we obtain the divergent and finite parts of $\int_\Phi Y_0$ by replacing $a^{2\epsilon}$ with $U^{-2\epsilon}$ in those terms whose integrals diverge as $U\to\infty$, and with 1 in the other terms. Thus,

$$\int_\Phi Y_0 \;=\; \frac{\not{n}^*}{2}\int_0^1\! dV\,\left[\frac{(V-V^2)^{-\epsilon}}{\epsilon(1+\epsilon)} + \frac{(V-V^2)^{\epsilon}}{\epsilon(1+\epsilon)} - \frac{(V-V^2)^{\epsilon}}{\epsilon(1-\epsilon)}\right] \;+\; O(\epsilon)\,,$$
$$=\;\frac{\not{n}^*}{2}\left(\frac{1}{\epsilon}-1\right) \;+\; O(\epsilon)\,. \qquad \text{(D.13)}$$

The divergent part of $I_E[f]$ may be obtained by combining Eqs. (D.7), (D.11), and (D.13).